

Almost partitioning 2-coloured complete 3-uniform hypergraphs into two monochromatic tight or loose cycles

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Abstract

We show that for every $\eta > 0$ there exists an integer n_0 such that every 2-colouring of the 3-uniform complete hypergraph on $n \geq n_0$ vertices contains two disjoint monochromatic tight cycles of distinct colours that together cover all but at most ηn vertices. The same result holds if we replace tight cycles with loose cycles.

1 Introduction

A certain type of Ramsey problems concerned with covering all vertices of the host graph instead of finding a small subgraph has gained popularity in recent years, both for graphs and hypergraphs. In particular, the problem of partitioning an edge-coloured complete (hyper-)graph into monochromatic cycles has received much attention. The recent surveys [5, 6] are a good starting point for the vast literature on the subject.

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Central to the area is an old conjecture of Lehel for graphs (see [2]). It states that for every n , every two-colouring of the edges of the complete graph K_n admits a covering of the vertices of K_n with at most two monochromatic vertex disjoint cycles of different colours. (For technical reasons, a single vertex or an edge count as a cycle.) This was confirmed for large n in [13, 1], and for all n by Bessy and Thomassé [3]. There is plenty of activity in determining the number of monochromatic cycles needed if K_n is coloured with more than two colours. It is known that this number is independent of n , but the best known lower and upper bounds, $r + 1$ and $100r \log r$, respectively, leave a considerable gap [15, 7].

For hypergraphs, much less is known. The problems transforms in the obvious way to hypergraphs, considering r -edge-colourings of the k -uniform $\mathcal{K}_n^{(k)}$ on n vertices, the only question is which type of hypergraph cycles we would like to work with. Referring to [5] for other results, we concentrate here on loose and tight cycles (see the next section for their definitions).

For loose cycles the problem was studied in [8, 17] and the best bound, due to Sárközy [17], shows that at most $50rk \log(rk)$ disjoint loose monochromatic cycles are needed for partitioning $\mathcal{K}_n^{(k)}$, a number that this is independent of n . In general even for the case $k = 3$ the problem is wide open and it is for example unknown whether one can cover almost all vertices with two disjoint monochromatic loose cycles.

Concerning the more restrictive notion of tight cycles, the situation is even worse and to our best knowledge, nothing is known. Gyárfás [6] conjectures that there is $c = c(r, k)$ such that every r -coloured $\mathcal{K}_n^{(k)}$ has a partition into at most c monochromatic tight cycles, but this is open even for the ‘easiest’ case of 3-uniform hypergraphs and two colours. Our main result establishes an approximate version for this case.

Theorem 1.1. *For every $\eta > 0$ there exists n_0 such that if $n \geq n_0$ then every two-coloring of the edges of the complete 3-uniform hypergraph $\mathcal{K}_n^{(3)}$ admits two vertex-disjoint monochromatic tight cycles, of distinct colours, which cover all but at most ηn vertices. Moreover, we can choose the parity of the length of each of the cycles.*

We might be interested in choosing the parity of the cycles for the following reason. If ℓ is even, then any 3-uniform tight cycle on ℓ edges contains a loose cycle. Hence, we can deduce that an analogue of Theorem 1.1 holds for loose cycles.

Corollary 1.2. *For every $\eta > 0$ there exists n_0 such that if $n \geq n_0$ then every two-coloring of the edges of the complete 3-uniform hypergraph $\mathcal{K}_n^{(3)}$*

admits two vertex-disjoint monochromatic loose cycles, of distinct colours, which cover all but at most ηn vertices.

We believe that the error term ηn in Theorem 1.1 can be improved and that every two-colouring of the edges of $\mathcal{K}_n^{(3)}$ admits two disjoint monochromatic tight cycles which cover all but at most a constant number c of vertices (for some c independent of n).

The proof of Theorem 1.1 is inspired by the work of Haxell et al. [10, 11] and relies on an application of the hypergraph regularity lemma [4]. This reduces the problem at hand to that of finding, in any two-colouring of the edges of an almost complete 3-uniform hypergraph, two disjoint *monochromatic connected matchings* which cover almost all vertices.

Here, as usual, a *matching* \mathcal{M} in hypergraph \mathcal{H} is a set of pairwise disjoint edges and $\mathcal{M} \subset \mathcal{H}$ is called *connected* if between every pair $e, f \in \mathcal{M}$ there is a *pseudo-path* in \mathcal{H} connecting e and f , that is, there is a sequence (e_1, \dots, e_p) of not necessarily distinct edges of \mathcal{H} such that $e = e_1, f = e_p$ and $|e_i \cap e_{i+1}| = 2$ for each $i \in [p-1]$. (Note that these pseudo-paths may use vertices outside $V(\mathcal{M})$.) Now, we call a connected matching \mathcal{M} in a 2-coloured hypergraph a *monochromatic connected matching* if all edges in \mathcal{M} and all edges on the connecting paths have the same colour.

So, our main contribution reduces to the following result, which might be of independent interest.

Theorem 1.3. *For every $\gamma > 0$ there is t_0 such that the following holds for every 3-uniform hypergraph \mathcal{H} with $t > t_0$ vertices and $(1 - \gamma)\binom{t}{3}$ edges. Any two-colouring of the edges of \mathcal{H} admits two disjoint monochromatic connected matchings covering at least $(1 - 240\gamma^{\frac{1}{6}})t$ vertices of \mathcal{H} .*

We prove Theorem 1.3 in Section 2. In Section 3, we introduce the regularity lemma for hypergraphs and state an embedding result from [11]. The proof of Theorem 1.1 will be given in Section 4.

2 Monochromatic connected matchings

Before giving the proof of Theorem 1.3 we introduce some notation and auxiliary results.

Let \mathcal{H} denote a k -uniform hypergraph, that is, a pair $\mathcal{H} = (V, E)$ with finite vertex set $V = V(\mathcal{H})$ and edge set $E = E(\mathcal{H}) \subset \binom{V}{k}$, where $\binom{V}{k}$ denotes the set of all k -element sets of V . Often \mathcal{H} will be identified with its edges, that is, $\mathcal{H} \subset \binom{V}{k}$ and for an edge $\{x_1, \dots, x_k\} \in \mathcal{H}$ we often omit brackets and write $x_1 \dots x_k$ only. A k -uniform hypergraph \mathcal{C} is called an

ℓ -cycle if there is a cyclic ordering of the vertices of \mathcal{C} such that every edge consists of k consecutive vertices, every vertex is contained in an edge and two consecutive edges (where the ordering of the edges is inherited by the ordering of the vertices) intersect in exactly ℓ vertices. For $\ell = 1$ we call the cycle *loose* whereas the cycle is called *tight* if $\ell = k - 1$ (and we do not consider other values of ℓ).

A *tight path* is a cycle from which one vertex and all incident edges are deleted. The *length* of a path, a pseudo-path or a cycle is the number of edges it contains. As above, two edges in \mathcal{H} are connected if there is a pseudo-path connecting them. Connectedness is an equivalence relation on the edge set of \mathcal{H} and the equivalence classes are called *connected components*.

All hypergraphs \mathcal{H} considered from now on are 3-uniform. We will need the following result concerning the existence of perfect matchings in 3-uniform hypergraphs with high minimum vertex degree.

Theorem 2.1 ([9]). *For all $\eta > 0$ there is a $n_0 = n_0(\eta)$ such that for all $n > n_0$, $n \in 3\mathbb{Z}$, the following holds. Suppose \mathcal{H} is 3-uniform hypergraph on n vertices such that every vertex is contained in at least $(\frac{5}{9} + \eta) \binom{n}{2}$ edges. Then \mathcal{H} contains a perfect matching.*

Denote by $\partial\mathcal{H}$ the *shadow* of \mathcal{H} , that is, the set of all pairs xy for which there exists z such that $xyz \in \mathcal{H}$. For a vertex x in a hypergraph \mathcal{H} , let $N_{\mathcal{H}}(x) = \{y : xy \in \partial\mathcal{H}\}$. For two vertices x, y , let $N_{\mathcal{H}}(x, y) = \{z : xyz \in \mathcal{H}\}$. Note that if $y \in N_{\mathcal{H}}(x)$ (equivalently, $x \in N_{\mathcal{H}}(y)$) then $N_{\mathcal{H}}(x, y) \neq \emptyset$. We call all such pairs xy of vertices *active*.

Lemma 2.2 ([10], Lemma 4.1). *Let $\gamma > 0$ and let \mathcal{H} be a 3-uniform hypergraph on $t_{\mathcal{H}}$ vertices and at least $(1 - \gamma) \binom{t_{\mathcal{H}}}{3}$ edges. Then \mathcal{H} contains a subhypergraph \mathcal{K} on $t_{\mathcal{K}} \geq (1 - 10\gamma^{1/6})t_{\mathcal{H}}$ vertices such that every vertex x of \mathcal{K} is in an active pair of \mathcal{K} and for all active pairs xy we have $|N_{\mathcal{K}}(x, y)| \geq (1 - 10\gamma^{1/6})t_{\mathcal{K}}$.*

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. For given $\gamma > 0$ let $\delta = 10\gamma^{1/6}$ and apply Theorem 2.1 with $\eta = 1/9$ to obtain n_0 . We choose $t_0 = \max\{\frac{2}{\delta}, \frac{n_0}{27\delta}\}$.

Suppose we are given a two-coloured 3-uniform hypergraph $\mathcal{H} = \mathcal{H}_{\text{red}} \cup \mathcal{H}_{\text{blue}}$ on $t_{\mathcal{H}} > t_0$ vertices and $(1 - \gamma) \binom{t_{\mathcal{H}}}{3}$ edges. Apply Lemma 2.2 to \mathcal{H} with parameter γ to obtain \mathcal{K} , $t := t_{\mathcal{K}}$ with the properties stated in the lemma. We wish to find two monochromatic connected matchings covering all but at most $24\delta t \leq 24\delta t_{\mathcal{H}}$ vertices of \mathcal{K} .

Since every vertex is in an active pair in \mathcal{K} , we have

$$|N_{\mathcal{K}}(x)| \geq (1 - \delta)t \quad \text{for all } x \in V(\mathcal{K}). \quad (1)$$

Let $\mathcal{K} = \mathcal{K}_{\text{red}} \cup \mathcal{K}_{\text{blue}}$ be the colouring of \mathcal{K} inherited from \mathcal{H} . Then a monochromatic component \mathcal{C} of \mathcal{K} is a connected component \mathcal{K}_{red} or $\mathcal{K}_{\text{blue}}$.

Observation 2.3 ([11], Observation 8.2). *For every vertex $x \in V(\mathcal{K})$ there exists a monochromatic component \mathcal{C}_x such that $|N_{\mathcal{C}_x}(x)| \geq (1 - \delta)t$.*

For each $x \in V(\mathcal{K})$ choose arbitrarily one component \mathcal{C}_x as in Observation 2.3. Let $R = \{x \in V(\mathcal{K}) : \mathcal{C}_x \text{ is red}\}$ and $B = \{x \in V(\mathcal{K}) : \mathcal{C}_x \text{ is blue}\}$, and note that these two sets partition $V(\mathcal{K})$.

Observation 2.4 ([11], Observation 8.4). *If $|R| \geq 6\delta t$ (or $|B| \geq 6\delta t$, respectively), then there is a red component \mathcal{R} (a blue component \mathcal{B}) such that $\mathcal{C}_x = \mathcal{R}$ ($\mathcal{C}_x = \mathcal{B}$) for all but at most $2\delta t$ vertices $x \in R$ ($x \in B$).*

Set $V_{\text{red}} := \{x \in R : \mathcal{C}_x = \mathcal{R}\}$ if $|V_{\text{red}}| \geq 6\delta t$, and set $V_{\text{blue}} := \{x \in B : \mathcal{C}_x = \mathcal{B}\}$ if $|B| \geq 6\delta t$. Otherwise let V_{red} , or V_{blue} , respectively, be the empty set. Our aim is to find two differently coloured disjoint connected matchings in \mathcal{K} that together cover all but $12\delta t \leq 24\delta t - 12\delta t$ vertices of $V_{\text{red}} \cup V_{\text{blue}}$.

We start by choosing a connected matching of maximal size in $\mathcal{R} \cup \mathcal{B}$. This matching decomposes into two disjoint monochromatic connected matchings, $\mathcal{M}_{\text{red}} \subset \mathcal{R}$ and $\mathcal{M}_{\text{blue}} \subset \mathcal{B}$, which together cover as many vertices as possible. Let $V'_{\text{red}} = V_{\text{red}} \setminus V(\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}})$ and $V'_{\text{blue}} = V_{\text{blue}} \setminus V(\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}})$. We may assume that V'_{red} or V'_{blue} has at least $12\delta t$ vertices, as otherwise we are done. By symmetry we may assume that

$$|V'_{\text{red}}| \geq 12\delta t. \quad (2)$$

Observe that there is no edge xy with $x \in V'_{\text{red}}$ and $y \in V'_{\text{blue}}$ such that $xy \in \partial\mathcal{R} \cap \partial\mathcal{B}$. Indeed, any such edge xy constitutes an active pair (by Lemma 2.2) and as $|V'_{\text{red}}| > \delta t + 2$, there must be a vertex $z \in V'_{\text{red}}$ such that xyz is an edge of \mathcal{K} . This contradicts the maximality of the matching $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$.

Next, we claim that

$$|V'_{\text{blue}}| \leq 2\delta t. \quad (3)$$

Assume otherwise. Then, Observation 2.3 and the choice of the set V_{red} implies that the number of edges between V'_{red} and V'_{blue} that belong to $\partial\mathcal{R}$ is at least

$$|V'_{\text{red}}| \cdot (|V'_{\text{blue}}| - \delta t) \geq \frac{1}{2}|V'_{\text{red}}| \cdot |V'_{\text{blue}}|.$$

Similarly, there are at least $|V'_{\text{blue}}| \cdot (|V'_{\text{red}}| - \delta t) > \frac{1}{2} |V'_{\text{red}}| \cdot |V'_{\text{blue}}|$ edges between V'_{red} and V'_{blue} that belong to $\partial \mathcal{B}$. As there is no edge xy with $x \in V'_{\text{red}}$ and $y \in V'_{\text{blue}}$ such that $xy \in \partial \mathcal{R} \cap \partial \mathcal{B}$, we have more than $|V'_{\text{red}}| \cdot |V'_{\text{blue}}|$ edges from V'_{red} to V'_{blue} . This yields a contradiction and (3) follows.

Because of the maximality of $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$, each edge having all its vertices in V'_{red} is blue. Fix one such edge xyz . Obtain V''_{red} from V'_{red} by deleting the at most δt vertices w with $wx \notin \partial \mathcal{R}$. Consider any edge $x'y'z'$ with $x', y', z' \in V''_{\text{red}}$. As the pairs $xy, xx', x'y'$ are all active, and $|V''_{\text{red}}| > 3\delta t$, there is a vertex $v \in V''_{\text{red}}$ that forms an edge with each of the three pairs, thus giving a pseudo-path in $\mathcal{K}[V''_{\text{red}}]$ from xyz to $x'y'z'$. Denote by \mathcal{B}'' the blue component of $\mathcal{K}[V''_{\text{red}}]$ that contains xyz , and let \mathcal{B}' be obtained from \mathcal{B}'' by deleting at most 2 vertices and all incident edges, so that $|V[\mathcal{B}']|$ is a multiple of 3. Then, by (2), we have

$$|V[\mathcal{B}']| \geq |V'_{\text{red}}| - \delta t - 2 \geq 10\delta t. \quad (4)$$

Let $x \in V[\mathcal{B}']$ be given. At least $|V[\mathcal{B}']| - \delta t$ vertices $y \in V[\mathcal{B}']$ are such that $xy \in \partial \mathcal{R}$, and, for each such y there are at least $|V[\mathcal{B}']| - \delta t$ vertices $z \in V[\mathcal{B}']$ such that $xyz \in \mathcal{B}'$. So, the total number of hyperedges of \mathcal{B}' that contain x is at least

$$\frac{1}{2}(|V[\mathcal{B}']| - \delta t)^2 \geq \frac{1}{2} \left(\frac{9}{10} |V[\mathcal{B}']| \right)^2 \geq \frac{2}{3} \binom{|V[\mathcal{B}']|}{2}.$$

Thus, Theorem 2.1 with $\eta = \frac{1}{9}$ yields a perfect matching $\mathcal{M}'_{\text{blue}}$ of \mathcal{B}' .

At this point, we have three disjoint monochromatic connected matchings, one in red ($\mathcal{M}_{\text{red}} \subset \mathcal{R}$) and two in blue ($\mathcal{M}_{\text{blue}} \subset \mathcal{B}$ and $\mathcal{M}'_{\text{blue}} \subset \mathcal{B}'$). Together, these matchings cover all but at most $3\delta t + 2$ vertices of $V_{\text{red}} \cup V_{\text{blue}}$ (by (3) and by (4)). In particular, we can assume that $\mathcal{M}_{\text{blue}}$ contains at least two hyperedges, as otherwise we can just forget about $\mathcal{M}_{\text{blue}}$ and are done.

Our aim is now to dissolve the blue matching $\mathcal{M}_{\text{blue}}$, and cover its vertices by new red edges, leaving at most $6\delta t$ vertices uncovered. In order to do so, let us first understand where the edges of $\mathcal{M}_{\text{blue}}$ lie.

For convenience, let us call an edge in \mathcal{K} *good* if two different pairs of its vertices $\{a, b\}$ and $\{c, d\}$ are such that $ab \in \partial \mathcal{R}$ and $cd \in \partial \mathcal{B}$. Notice that every good red edge is contained in \mathcal{R} and every good blue edge is contained in \mathcal{B} .

First, we claim that for every edge $uvw \in \mathcal{M}_{\text{blue}}$,

$$|\{u, v, w\} \cap V_{\text{blue}}| \leq 1. \quad (5)$$

Indeed, otherwise there is an edge $uvw \in \mathcal{M}_{\text{blue}}$ with $u, v \in V_{\text{blue}}$. By the definition of \mathcal{B} , and by (2), there is an active edge $ua \in \partial\mathcal{B}$ with $a \in V'_{\text{red}}$. As ua is an active pair, as a has very large degree in $\partial\mathcal{R}$, and by (2), there is an edge $uab \in \mathcal{K}$ with $b \in V'_{\text{red}}$ such that $ab \in \partial\mathcal{R}$. Hence uab is a good edge. Similarly, there is a good edge vcd , with $c, d \in V'_{\text{red}} \setminus \{a, b\}$. Remove the edge uvw from $\mathcal{M}_{\text{blue}}$ and add edges uab and vcd to either \mathcal{M}_{red} or $\mathcal{M}_{\text{blue}}$, according to their colour. The resulting matching covers more vertices than the matching $M_A \cup \mathcal{M}_{\text{blue}}$, a contradiction. This proves (5).

Next, we claim that there is no edge $uvw \in \mathcal{M}_{\text{blue}}$ with

$$|\{u, v, w\} \cap V_{\text{blue}}| = 1. \quad (6)$$

Assume otherwise. Then there is an edge $uvw \in \mathcal{M}_{\text{blue}}$ with $u \in V_{\text{blue}}$ and $v, w \in V_{\text{red}}$. As in the proof of (5), we can cover u with a good edge uab such that $a, b \in V'_{\text{red}}$. Moreover, since vw is an active pair, and v has very large degree in $\partial\mathcal{R}$, there is an edge vwc with $c \in V'_{\text{red}} \setminus \{a, b\}$ and $cv \in \partial\mathcal{R}$. Since $vw \in \partial\mathcal{B}$, the edge vwc is good. So we can remove uvw from $\mathcal{M}_{\text{blue}}$ and add edges uab and vwc to $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$, thus covering three additional vertices. This gives the desired contradiction to the choice of $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$, and proves (6).

Putting (5) and (6) together, we know that for every edge $uvw \in \mathcal{M}_{\text{blue}}$ we have $u, v, w \in V_{\text{red}}$. Consider any two edges $u_1v_1w_1, u_2v_2w_2 \in \mathcal{M}_{\text{blue}}$. As before, there are vertices $a, b \in V'_{\text{red}}$ such that edges v_1w_1a, v_2w_2b are good. Now, if there is a red edge u_1u_2c with $c \in V'_{\text{red}}$ and $u_1c \in \partial\mathcal{R}$ then we can remove edges $u_1v_1w_1, u_2v_2w_2$ and add the red edge u_1u_2c to \mathcal{M}_{red} and edges v_1w_1a, v_2w_2b to $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$, according to their colour, contradicting the choice of $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$. Therefore, for any choice of $u_1v_1w_1, u_2v_2w_2 \in \mathcal{M}_{\text{blue}}$, we have that

$$\text{all edges } u_1u_2c \text{ with } c \in V'_{\text{red}} \text{ and } u_1c \in \partial\mathcal{R} \text{ are blue.} \quad (7)$$

Moreover, if there is a blue edge u_1u_2x with $x \in \{v_1, w_1, v_2, w_2\}$ then u_1u_2 is an active pair. In that case, we can calculate as before that an edge u_1u_2c with $c \in V'_{\text{red}}$ and $u_1c \in \partial\mathcal{R}$ exists, and by (7), this edge is blue. The existence of the blue edge u_1u_2x implies that we can link u_1u_2c to $\mathcal{M}_{\text{blue}}$ with a blue tight path. Thus, removing $u_1v_1w_1$ and $u_2v_2w_2$ from $\mathcal{M}_{\text{blue}}$ and adding $v_1w_1a, v_2w_2b, u_1u_2c$ to $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$ (where a, b are as above), we obtain a contradiction to the choice of $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$. So, for any choice of $u_1v_1w_1, u_2v_2w_2 \in \mathcal{M}_{\text{blue}}$, we have that

$$\text{all edges } u_1u_2x \text{ with } x \in \{v_1, w_1, v_2, w_2\} \text{ are red.} \quad (8)$$

We can now dissolve the edges of $\mathcal{M}_{\text{blue}}$. For this, separate each hyperedge uvw in $\mathcal{M}_{\text{blue}}$ into an edge uv and a single vertex w . Let X be the set of all edges uv , and let Y be the set of all vertices w obtained in this way. Note that every $uv \in X$ is an active pair in \mathcal{K} , and therefore forms a hyperedge uvw' with all but at most δt of the vertices $w' \in Y$. Moreover, all but at most δt of these hyperedges uvw' are such that $uw' \in \partial\mathcal{R}$, because of the large degree u has in $\partial\mathcal{R}$.

Consider the bipartite graph on $X \cup Y$ which has an edge between $uv \in X$ and w' whenever the hyperedge uvw' exists in \mathcal{K} and $uw' \in \partial\mathcal{R}$. Then for each $X' \subseteq X$ we have that $|N(X')| \geq |Y| - 2\delta t \geq |X'| - 2\delta t$. So by the defect form of Hall's Theorem, there is a matching covering all but at most $2\delta t$ vertices of $|X'|$. In \mathcal{K} , this corresponds to a matching $\mathcal{M}'_{\text{red}}$ covering all but at most $6\delta t$ vertices of $V(\mathcal{M}_{\text{blue}})$. By (8), all hyperedges of $\mathcal{M}'_{\text{red}}$ are red. Furthermore, since we ensured that every hyperedge in \mathcal{M}_{red} contains a pair uw' that forms an edge of $\partial\mathcal{R}$, we know that \mathcal{M}_{red} and $\mathcal{M}'_{\text{red}}$ belong to the same red component of \mathcal{K} . In other words, $\mathcal{M}_{\text{red}} \cup \mathcal{M}'_{\text{red}}$ and $\mathcal{M}'_{\text{blue}}$ are the two monochromatic connected matchings we had to find. \square

3 Hypergraph regularity

In this section we introduce the regularity lemma for 3-uniform hypergraphs and state an embedding result from [11].

Graph regularity. Let G be a graph G and let $X, Y \subseteq V(G)$ be disjoint. The *density* of (X, Y) is $d_G(X, Y) = \frac{e_G(X, Y)}{|X||Y|}$ where $e_G(X, Y)$ denotes the number of edges of G between X and Y .

The bipartite graph G on the partition classes X and Y is called (d, ε) -*regular*, if $|d_G(X', Y') - d| < \varepsilon$ holds for all $X' \subseteq X$ and $Y' \subseteq Y$ of size $|X'| > \varepsilon|X|$ and $|Y'| > \varepsilon|Y|$. If $d = d_G(X, Y)$ we say that G is ε -*regular*.

Hypergraph regularity. Let \mathcal{H} be a 3-uniform hypergraph. Let $P = P^{12} \cup P^{13} \cup P^{23}$ with $V(P) \subset V(\mathcal{H})$ be a tripartite graph which we also refer to as *triad*. By $\mathcal{T}(P)$ denote the 3-uniform hypergraph on $V(P)$ whose edges are the triangles of P . The *density of \mathcal{H} with respect to P* is

$$d_{\mathcal{H}}(P) = \frac{|\mathcal{H} \cap \mathcal{T}(P)|}{|\mathcal{T}(P)|}.$$

Similarly, for a tuple $\vec{Q} = (Q_1, \dots, Q_r)$ of subgraphs of P , we define the *density of \mathcal{H} with respect to \vec{Q}* as

$$d_{\mathcal{H}}(\vec{Q}) = \frac{|\mathcal{H} \cap \bigcup_{i \in [r]} \mathcal{T}(Q_i)|}{|\bigcup_{i \in [r]} \mathcal{T}(Q_i)|}.$$

Let $\alpha, \delta > 0$ and let $r > 0$ be an integer. We say that \mathcal{H} is (α, δ, r) -*regular* with respect to P if, for every r -tuple $\vec{Q} = (Q_1, \dots, Q_r)$ of subgraphs of P satisfying $|\bigcup_{i \in [r]} \mathcal{T}(Q_i)| > \delta |\mathcal{T}(P)|$, we have $|d_{\mathcal{H}}(\vec{Q}) - \alpha| < \delta$. If $\alpha = d_{\mathcal{H}}(P)$ we say that \mathcal{H} is (δ, r) -*regular* with respect to P , and in the same situation, we say P is (δ, r) -*regular* (with respect to \mathcal{H}).

Moreover, if the bipartite graphs P^{12}, P^{13}, P^{23} of an (α, δ, r) -regular $P = P^{12} \cup P^{13} \cup P^{23}$ are $(1/\ell, \varepsilon)$ -regular then we say that the pair (\mathcal{H}, P) is an $(\alpha, \delta, \ell, r, \varepsilon)$ -*regular complex*.

Finally, a partition of V into $V_0 \cup V_1 \cup \dots \cup V_t$ is called an *equipartition* if $|V_0| < t$ and $|V_1| = |V_2| = \dots = |V_t|$.

We state the regularity lemma for 3-uniform hypergraphs [4] as presented in [14].

Theorem 3.1 (Regularity Lemma for 3-uniform Hypergraphs). *For all $\delta, t_0 > 0$, all integer-valued functions $r = r(t, \ell)$, and all decreasing sequences $\varepsilon(\ell) > 0$ there exist constants T_0, L_0 and N_0 such that every 3-uniform hypergraph \mathcal{H} with at least N_0 vertices admits a vertex equipartition*

$$V(\mathcal{H}) = V_0 \cup V_1 \cup \dots \cup V_t \quad \text{with } t_0 \leq t < T_0,$$

and, for each pair i, j , $1 \leq i < j \leq t$, an edge partition of the complete bipartite graph

$$K(V_i, V_j) = \bigcup_{k \in [\ell]} P_k^{ij} \quad \text{with } 1 \leq \ell < L_0$$

such that

1. *all graphs P_k^{ij} are $(1/\ell, \varepsilon(\ell))$ -regular.*
2. *\mathcal{H} is (δ, r) -regular with respect to all but at most $\delta \ell^3 t^3$ tripartite graphs $P_a^{hi} \cup P_b^{hj} \cup P_c^{ij}$.*

Note that the same partitions satisfy the conclusions of Theorem 3.1 for the complement of \mathcal{H} as well. Further, as noted in [11] by choosing a random index $k_{ij} \in [\ell]$ for each pair (V_i, V_j) Markov's inequality yields that with positive probability there are less than $2\delta t^3$ chosen triads which fail to be (δ, r) -regular. Hence one obtains the following.

Observation 3.2. *In the partition produced by Theorem 3.1 there is a family \mathcal{P} of bipartite graphs $P^{ij} = P_{k_{ij}}^{ij}$ with vertex classes V_i, V_j , where $1 \leq i < j \leq t$, such that \mathcal{H} is (δ, r) -regular with respect to all but at most $2\delta t^3$ tripartite graphs $P^{hi} \cup P^{hj} \cup P^{ij}$.*

We end this section with a result from [16] and [11] which allows embedding tight paths in regular complexes. In the following, an S -avoiding tight path is one which does not contain any vertex from S . (Note that although Lemma 4.6 from [11] is stated slightly differently, its proof actually yields the version below.)

Lemma 3.3 ([11], Lemma 4.6). *For each $\alpha \in (0, 1)$ there exist $\delta_1 > 0$ and sequences $r(\ell)$, $\varepsilon(\ell)$, and $n_1(\ell)$, for $\ell \in \mathbb{N}$, with the following property. For each $\ell \in \mathbb{N}$, and each $\delta \leq \delta_1$, if (\mathcal{H}, P) is a $(d_{\mathcal{H}}(P), \delta, \ell, r(\ell), \varepsilon(\ell))$ -complex with $d_{\mathcal{H}}(P) \geq \alpha$ and all of the three vertex classes of P have the same size $n > n_1(\ell)$, then there is a subgraph P_0 on at most $27\sqrt{\delta}n^2/\ell$ edges of P such that, for all ordered pairs of disjoint edges $(e, f) \in (P \setminus P_0) \times (P \setminus P_0)$ there is $m = m(e, f) \in [3]$ such that the following holds. For every $S \subseteq V(\mathcal{H}) \setminus (e \cup f)$ with $|S| < n/(\log n)^2$, and for each ℓ with $3 \leq \ell \leq (1 - 2\delta^{\frac{1}{4}})n$, there is an S -avoiding tight path from e to f of length $3\ell + m$ in \mathcal{H} .*

4 Proof of Theorem 1.1

We follow a procedure suggested by Łuczak in [12] for graphs and used for tight cycles in 3-uniform hypergraphs in [11].

Proof of Theorem 1.1. For given $\eta > 0$ we apply Lemma 1.3 with $\gamma = (\eta/480)^6$ to obtain t_0 . With foresight apply Lemma 3.3 with $\alpha = 1/2$ to obtain δ_1 , and sequences $r(\ell)$, $\varepsilon(\ell)$, and $n_1(\ell)$. Finally, apply Theorem 3.1 with t_0 , $r(t, \ell) = r(\ell)$, $\varepsilon(\ell)$, $n_1(\ell)$ and $\delta = \min\{\delta_1/2, \gamma/48, (\eta/16)^4\}$ to obtain constants T_0 , L_0 and N_0 .

Given a two-colouring $\mathcal{K}_n = \mathcal{H}_{\text{red}} \cup \mathcal{H}_{\text{blue}}$ of the 3-uniform complete hypergraph \mathcal{K}_n on $n > N_0$ vertices. Apply Theorem 3.1 with the chosen constants to \mathcal{H}_{red} to obtain partitions

$$V(\mathcal{K}_n) = V_0 \cup V_1 \cup \dots \cup V_t \quad \text{and} \quad K(V_i, V_j) = \bigcup_{k \in [\ell]} P_k^{ij}, 1 \leq i < j \leq t$$

with $t_0 \leq t < T_0$, and $\ell < L_0$ which satisfy the properties detailed in Theorem 3.1. The partitions satisfy the same properties for $\mathcal{H}_{\text{blue}}$ as it is the complement hypergraph of \mathcal{H}_{red} .

Observation 3.2 then yields a family of $(1/\ell, \varepsilon)$ -regular bipartite graphs $P^{ij} = P_{k_{ij}}^{ij}$, one for each pair (V_i, V_j) , $1 \leq i < j \leq t$, such that \mathcal{H}_{red} (and thus also $\mathcal{H}_{\text{blue}}$) is (δ, r) -regular with respect to all but at most $2\delta t^3$ tripartite graphs $P^{ijk} = P^{ij} \cup P^{ik} \cup P^{jk}$. We use this family to construct the reduced hypergraph \mathcal{R} which has the vertex set $[t]$ and the edge set consisting of all triples ijk such that P^{ijk} is (δ, r) -regular. Further, colour the edge ijk red if $d_{\mathcal{H}_{\text{red}}}(P^{ijk}) \geq 1/2$ and blue otherwise. Then \mathcal{R} has at least $\binom{t}{3} - 2\delta t^3 > (1 - \gamma)\binom{t}{3}$ edges and as $d_{\mathcal{H}_{\text{red}}}(P^{ijk}) + d_{\mathcal{H}_{\text{blue}}}(P^{ijk}) = 1$ we obtain a two-colouring of $\mathcal{R} = \mathcal{R}_{\text{red}} \cup \mathcal{R}_{\text{blue}}$.

Since $t \geq t_0$ Lemma 1.3 yields two disjoint monochromatic connected matchings \mathcal{M}_{red} and $\mathcal{M}_{\text{blue}}$ which cover all but at most $240\gamma^{\frac{1}{6}}t \leq \eta t/2$ vertices of \mathcal{R} and in what follows we will turn these connected matchings into disjoint monochromatic tight cycles in \mathcal{K}_n .

We start by choosing a red pseudo-path $\mathcal{Q}_{\text{red}} = (e_1, \dots, e_p) \subset \mathcal{R}_{\text{red}}$ which contains the matching \mathcal{M}_{red} . This is possible since \mathcal{M}_{red} is a connected matching, and so, consecutive matching edges $g_s, g_{s+1} \in \mathcal{M}_{\text{red}}$ are connected by a red pseudo-path of length at most $\binom{t}{3}$. The concatenation of these paths then forms a \mathcal{Q}_{red} as desired. In the same manner, choose a blue pseudo-path $\mathcal{Q}_{\text{blue}} = (e'_1, \dots, e'_q) \subset \mathcal{R}_{\text{blue}}$ containing the matching $\mathcal{M}_{\text{blue}}$. Note that although \mathcal{M}_{red} and $\mathcal{M}_{\text{blue}}$ are disjoint, the two paths \mathcal{Q}_{red} and $\mathcal{Q}_{\text{blue}}$ may have vertices in common.

For each edge $\{i, j, k\} = e_s \in \mathcal{Q}_{\text{red}}$ let P^s denote the triad $P^{ij} \cup P^{ik} \cup P^{jk}$ on the partition classes $V_i \cup V_j \cup V_k$ and recall that P^s is (δ, r) -regular (with respect to \mathcal{H}_{red}). Lemma 3.3 guarantees that one can find long tight paths in the complex $(\mathcal{H}_{\text{red}}, P^s)$ for each matching edge $e_s \in \mathcal{M}_{\text{red}} \subset \mathcal{Q}_{\text{red}}$ which exhaust almost all vertices of $V_i \cup V_j \cup V_k$. Using the connectedness of \mathcal{Q}_{red} and Lemma 3.3, we then want to connect these long tight paths by short tight paths, hence obtain a tight cycle \mathcal{C}_{red} which covers almost all vertices of \mathcal{K}_n spanned by \mathcal{M}_{red} . We wish to do the same with $\mathcal{Q}_{\text{blue}}$ to obtain a tight cycle $\mathcal{C}_{\text{blue}}$ which covers almost all vertices of \mathcal{K}_n spanned by $\mathcal{M}_{\text{blue}}$. The two cycles \mathcal{C}_{red} and $\mathcal{C}_{\text{blue}}$ then exhaust most of the vertices of \mathcal{K}_n .

To keep the two cycles disjoint, however, the strategy will be slightly less straightforward. First, we will find two disjoint short tight cycles $\mathcal{C}'_{\text{red}}$ and $\mathcal{C}'_{\text{blue}}$ in \mathcal{K}_n visiting all triads P^s corresponding to edges $e_s \in \mathcal{M}_{\text{red}}$ and P^s corresponding to $e'_s \in \mathcal{M}_{\text{blue}}$, respectively. Then, for each edge $e_s \in \mathcal{M}_{\text{red}}$, and each edge $e'_s \in \mathcal{M}_{\text{blue}}$, we will replace parts of the cycles $\mathcal{C}'_{\text{red}}$, and of $\mathcal{C}'_{\text{blue}}$, i.e., paths corresponding to e_s and e'_s by long tight paths as mentioned above. We now give the details of this idea.

For each $s = 1, \dots, p$, apply Lemma 3.3 to the complex $(\mathcal{H}_{\text{red}}, P^s)$ to

obtain the subgraph $P_0^s \subset P^s$ of “prohibited” edges and let

$$B_s = (P^s \setminus P_0^s) \cap (P^{s+1} \setminus P_0^{s+1})$$

which is a bipartite graph on the partition classes $V_i \cup V_j$ where $\{i, j\} = e_s \cap e_{s+1}$. We choose mutually distinct edges $f_s, g_s \in B_s$, $s \in [p-1]$ which is possible due to the restriction on $|P_0^s|$ provided n is sufficiently large. Using Lemma 3.3 we then find a short tight cycle $\mathcal{C}'_{\text{red}}$ by concatenating disjoint paths each of length at most 12 between f_1 and f_2 , f_2 and $f_3 \dots$ between f_{p-1} and g_{p-1} and backwards between g_{p-1} and $g_{p-2} \dots$ and finally between g_1 and f_1 . Note that the lemma allows the paths to be S -avoiding for any vertex set S of size $|S| < n' / (\log n')^2$ where n' is the size of the partition classes. Therefore, to guarantee the disjointness of the paths, we simply choose S to be the vertices of the paths constructed so far which has size at most $24p$, i.e., independent of $n' > n/2t$. In the same way choose a short tight cycle $\mathcal{C}'_{\text{blue}}$ disjoint from $\mathcal{C}'_{\text{red}}$ by including $V(\mathcal{C}'_{\text{red}})$ to S in the applications of Lemma 3.3.

Let $S' = V(\mathcal{C}'_{\text{red}}) \cup V(\mathcal{C}'_{\text{blue}})$ which satisfies $|S'| < n' / (\log n')^2$. It is easy to see that for each $e_s \in \mathcal{M}_{\text{red}}$ there are two non-prohibited edges in P^s , connected by a subpath of $\mathcal{C}'_{\text{red}}$ which is entirely contained in $(\mathcal{H}_{\text{red}}, P^s)$. Hence, by Lemma 3.3 we can replace this short path by an S -avoiding path in $(\mathcal{H}_{\text{red}}, P^s)$ which covers all but at most $4\delta^{1/4}n'$ vertices and having any desired parity. Doing this for all $e_s \in \mathcal{M}_{\text{red}}$ and all $e'_s \in \mathcal{M}_{\text{blue}}$ and noting that $n' \leq n/t$ we obtain two monochromatic disjoint tight cycles which cover all but at most

$$\begin{aligned} & (|\mathcal{M}_{\text{red}}| + |\mathcal{M}_{\text{blue}}|)4\delta^{1/4}n' + (|V(\mathcal{R})| - |\mathcal{M}_{\text{red}}| + |\mathcal{M}_{\text{blue}}|)n' + |V_0| \\ & \leq \frac{1}{4}\eta n + \frac{1}{2}\eta n + t \leq \eta n \end{aligned}$$

vertices of \mathcal{K}_n , and have any desired parity. This finishes the proof of the theorem. \square

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